Towards Computational UIP in Cubical Agda

MPRI M2 Internship Presentation

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Outline

Introduction

Uniqueness of Identity Proofs (UIP)

Equality in Dependent Type Theory

- Agda [Agd25a], Rocq [Roc25], Lean [MU21], etc. are based on Dependent Type Theory
  - Slogan: "propositions as types, proofs as programs"
- The equality proposition is represented by the **Martin-Löf Identity Type**
  - one constructor: reflexivity
  - eliminator J can derive symmetry, transitivity, and substitutivity.
- Also known as **propositional** equality, not to be confused with **definitional** equality (meta-theoretic equality).

Uniqueness of Identity Proofs (UIP)

Uniqueness of Identity Proofs

Uniqueness of Identity Proofs (UIP) [HS94] postulate: are proofs of equality in Type Theory unique?

- Setoid model [Hof97] supports UIP
- Groupoid model [HS94] refutes UIP

hence UIP does not necessarily hold for every type theory.

Homotopy Type Theory (HoTT) [Uni13]

Vastly generalises the non-uniqueness of identity proofs.

Slogan: "propositions as types, proofs as programs, equalities as paths"

Univalence axiom: $(A \equiv_{\mathrm{Type}} B) \simeq (A \simeq B)$ incompatible with UIP.

Cubical Type Theory & Cubical Agda

Cubical Type Theory (CubTT)

A flavour of HoTT [Coh+17] implemented by Cubical Agda [VMA21].

Some **advantages** of CubTT over Dependent Type Theory:

- Functional extensionality (pointwise equal functions are equal)
- Quotient Inductive Types (QITs) as an instance of Higher Inductive Types (HITs)

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Cubical Agda and... UIP?

Suppose if we have a *consistent* way to combine them, then we could get...

- a simpler system for verification (one equality level instead of *infinitely* many)
- a new metatheory which researchers would like to work in [Coc19, Pit20, Shu17]
- ...but naïvely postulating UIP in Cubical Agda **blocks** computation!

$$\ldots(\mathrm{UIP}\ (A \times A)\ a\ b\ p\ q)\ldots \to_\beta \ldots(\mathrm{UIP}\ (A \times A)\ a\ b\ p\ q)\ldots$$

$$\to_\beta^* \ldots(\mathrm{UIP}\ (A \times A)\ a\ b\ p\ q)\ldots \text{ never reduces!}$$

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- if we have a Cubical Agda variant **without Glue**, then one can safely postulate **UIP** as an axiom (consistent by a set model)...
- where functional extensionality holds, QITs too...
- What if: instead of a computation blocking axiom, we can have UIP that **computes**?

Computational UIP in Cubical Agda

Our plan for computational behaviour for UIP in Cubical Agda:

1. A variant of Cubical Agda **without** Glue Types (hence without univalence) to ensure UIP compatibility.
2. The proofs of UIP compute automatically based on their type derivation.

$$\begin{aligned} & \ldots(\mathrm{UIP}\ (A \times A)\ a\ b\ p\ q)\ldots \\ \to_\beta\ & \ldots[\text{UIP-product}\ (\mathrm{UIP}\ A\ (\pi_1 a)\ (\pi_1 b)\ldots)\ (\mathrm{UIP}\ A\ (\pi_2 a)\ (\pi_2 b)\ldots)]\ldots \quad \text{inductive case} \\ \to_\beta\ & \ldots \quad \text{computes away} \end{aligned}$$

   and so on, reaching base types (such as $\mathbb{0}, \mathbb{1}$) or quotient inductive types. Essentially a **proof by induction** on type derivation!

3. It remains to detail all computation rules (e.g. inductive cases: preservation by type formers) in a suitable UIP formulation.

Contributions

In this internship, I

1. extended Cubical Agda (which implements CubTT) with a `--cubical=no-glue` **variant** (https://github.com/agda/agda/pull/7861)

2. propose implementing **computational UIP as an induction** on type derivation
   - base cases = base types + (possibly higher) inductive types
   - inductive cases = preservation by type formers

3. propose **homogeneous** SqFill and **heterogeneous** SqPFill as equivalent generalisations of UIP

4. prove the preservation of SqFill and SqPFill 👏 by 4 type formers:
   - Pi (dependent functions)
   - Sigma (dependent products),
   - Coproducts (disjoint sum), and
   - Path types (equality types).

A (Pictorial) Crash Course on Cubical Type Theory

Cubical Type Theory (Simplified)

- CubTT takes the "equalities as **paths**" slogan literally:
  - **paths** (just like in topology) are functions from the interval type $p : I \to A$
    $p(0) = a, p(1) = b$, then $p : a \equiv b$
  - points (A), line ($I \to A$), squares ($I^2 \to A$), cubes ($I^3 \to A$), ...

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  - points (A), line ($I \to A$), squares ($I^2 \to A$), cubes ($I^3 \to A$), ...
- Two primitive structures:
  1. **Interval type I** (for paths): a de Morgan algebra
     (i.e. bounded distributed lattice $(0, 1, \wedge, \vee)$ + de Morgan involution $\sim$)
     - de Morgan laws (e.g. $\sim (a \wedge b) =\sim a \vee \sim b$)
     - distributivity laws (e.g. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$)
     - but **NO** LEM ($a \vee \sim a = 1$) nor absurdity ($a \wedge \sim a = 0$)
  2. Glue Types (for univalence)
- Two primitive operations in Cubical Agda (Kan operations):
  1. **hcomp** (composition) and
  2. **transp** (transport)

Pictorial interpretation (node = object, edge = path)

- hcomp: *homogeneously* composing a path with partial sides to get a “lid”.

$$a \xrightarrow[p]{} b$$

hcomp of a homogeneous path p along partial sides u.

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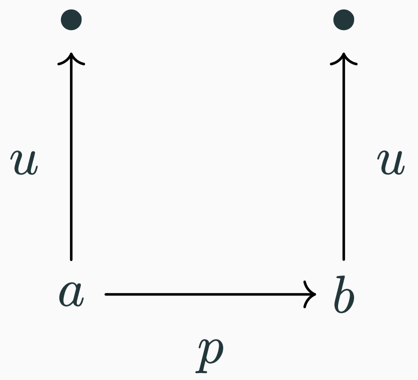

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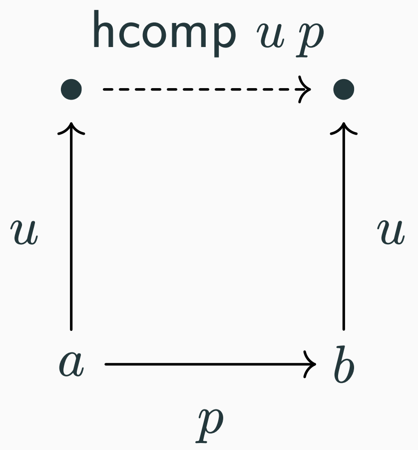

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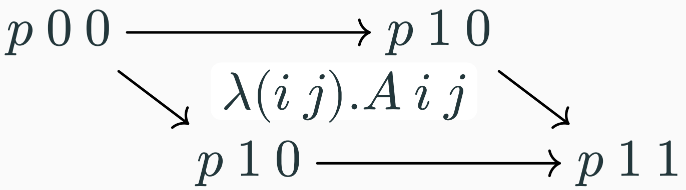

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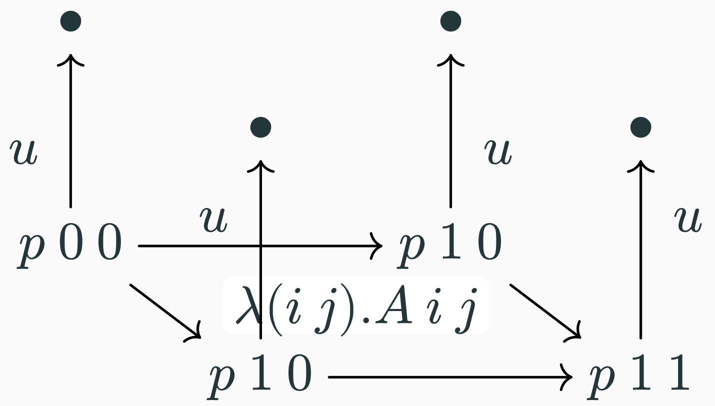

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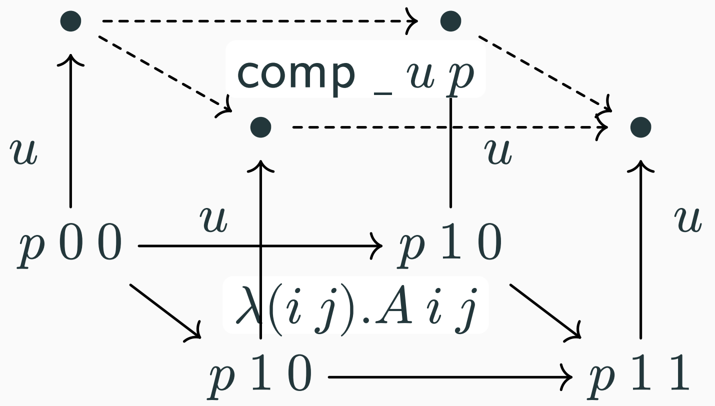

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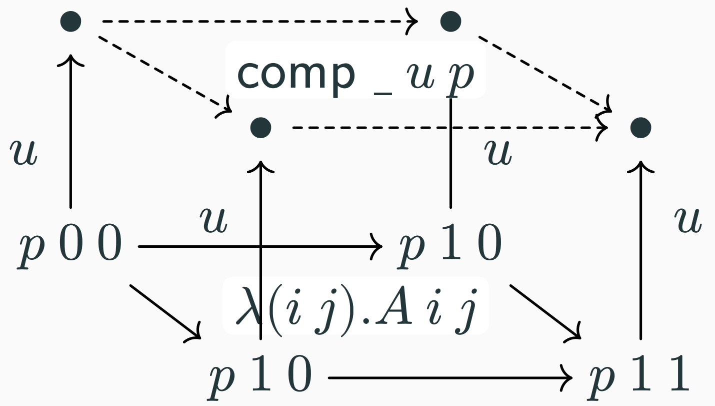

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$a : A\,0$

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$\uparrow$ transport-fill $A\,(p\,0)$

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- transport : $\forall (A : I \to \mathrm{Type}). A\,0 \to A\,1$.
  - "transportee" and the transported target are always propositionally equal.
  - In a square of types $A : I \to I \to \mathrm{Type}$, any two types $A\,i\,j$ and $A\,i'\,j'$ has a path between them: $\lambda(k : I). A\ \mathrm{coe}_k(i, i')\ \mathrm{coe}_k(j, j')$.

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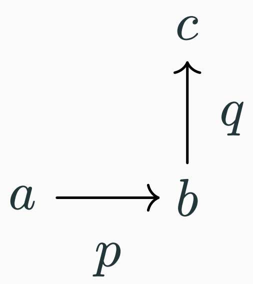

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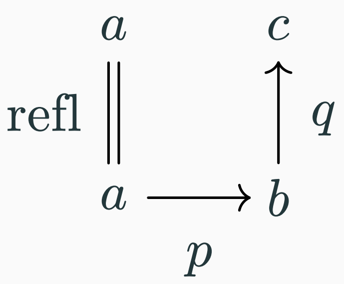

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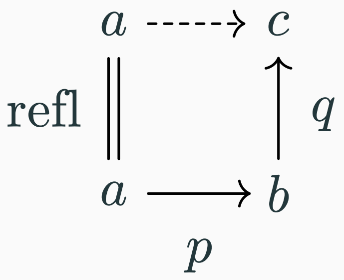

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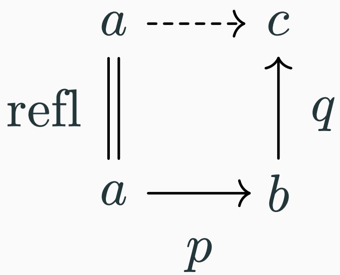

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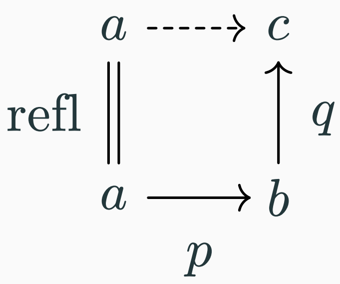

- substitutivity ($\forall(P : A \to \mathrm{Type}).(x\ y : A).(p : x = y).Px \to Py$)?

  transport along the line of types $\lambda(i : I).P(p\ i)$.

Implementation: Cubical Agda without Glue

Cubical Agda without Glue

- … is a Cubical Agda variant designed to be *compatible* with UIP.
- Three Cubical-related variants already exist:
  1. Full Cubical `--cubical` and
  2. Cubical with Erased Glue `--erased-cubical`
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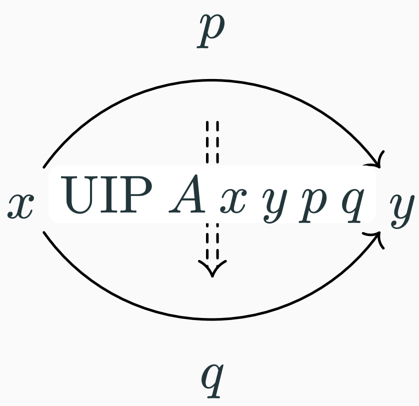
- https://github.com/agda/agda/pull/7861
- Type checks Glue-less parts of the Cubical Library [Agd25b]!
- Type checks our SqFill, SqPFill proofs! (which *definitely* shouldn't use Glue...)

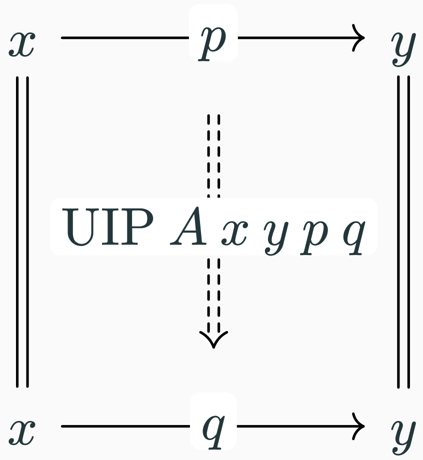
The Tale of Two Square-Fills

Generalising UIP: SqFill

- $\text{UIP}\ A$: for any two proofs of identity $p\ q : x \equiv_A y$, we have $p \equiv_{x \equiv_A y} q$.



- UIP A: for any square with two (opposing) reflexive sides, we have a filling.



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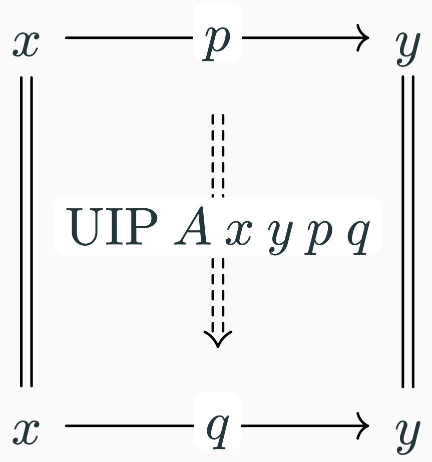

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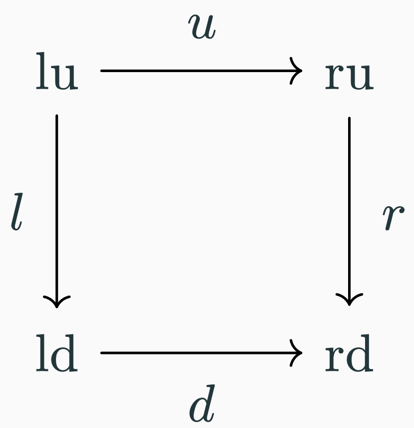

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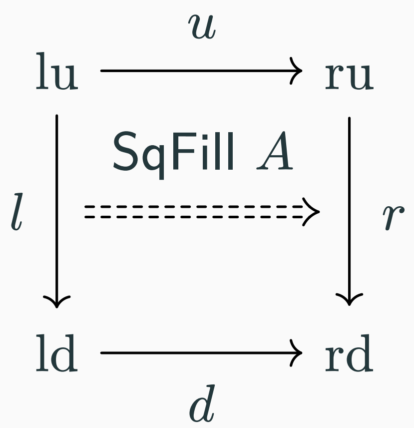

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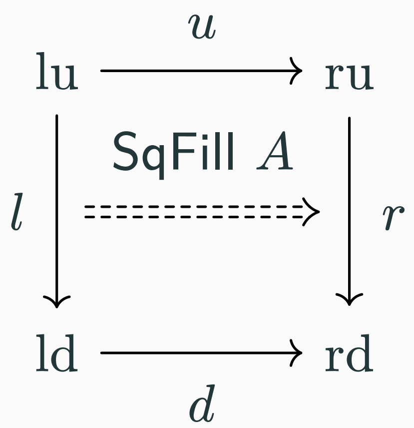

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- Let’s see how hard are the SqFill preservation proofs.

SqFill: Summary

	SqFill (*homogeneous* Square-Filling)
Pi	Trivial (no Kan operations)
Sigma	**Complicated**: transport-fill-align
Coproducts	Standard encode-decode proof (J, hcomp)
Path Types	Simple (a single hcomp)

How did the SqFill proofs go?

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$$
\begin{array}{ccc}
\mathrm{lu} & \xrightarrow{\quad u \quad} & \mathrm{ru} \\
l \downarrow & \Pi(a : A).B\, a & \downarrow r \\
\mathrm{ld} & \xrightarrow[\quad d \quad]{} & \mathrm{rd}
\end{array}
$$

Hollow square of functions given.

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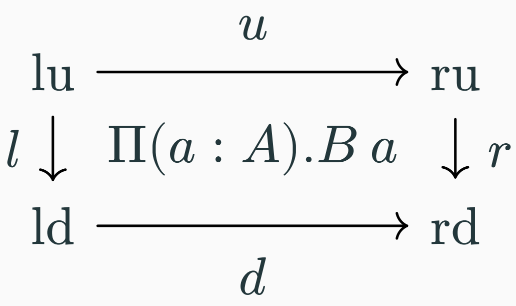

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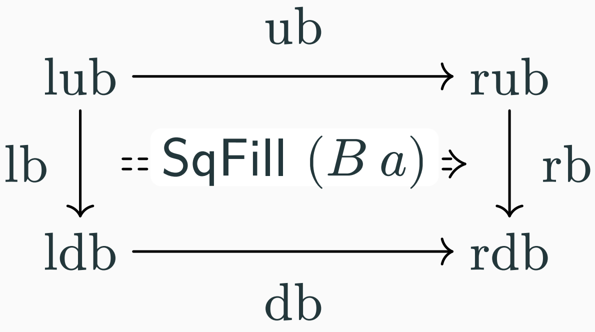

Applying the hollow square to a, and fill.

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- Theorem (SqFill-Pi): If $A : \mathrm{Type}$ and $B : A \to \mathrm{Type}$ has the SqFill property, then so does $\Pi(a : A).B\ a$. **Trivial!**

```
SqFillPiAB : SqFill ((a : A) → B a)
SqFillPiAB l r u d i j a = SqFillB a (λ i → l i a) (λ i → r i a) (λ i → u i a) (λ i → d i a) i j
```

How did the SqFill proofs go?

- Theorem (SqFill-Pi): If $A : \mathrm{Type}$ and $B : A \to \mathrm{Type}$ has the SqFill property, then so does $\Pi(a : A).B\ a$. **Trivial!**
- Theorem (SqFill-Sigma): If $A : \mathrm{Type}$ and $B : A \to \mathrm{Type}$ has the SqFill property, then so does $\Sigma(a : A).B\ a$.

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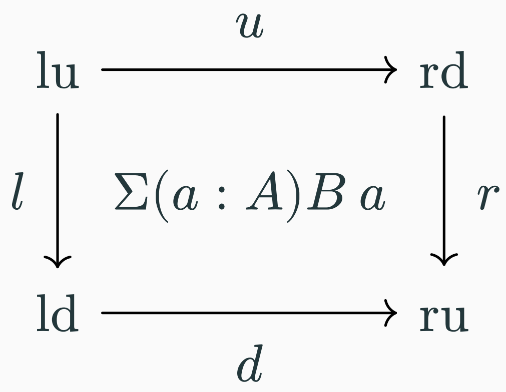

Hollow square in $\Sigma(a : A)B\ a$.

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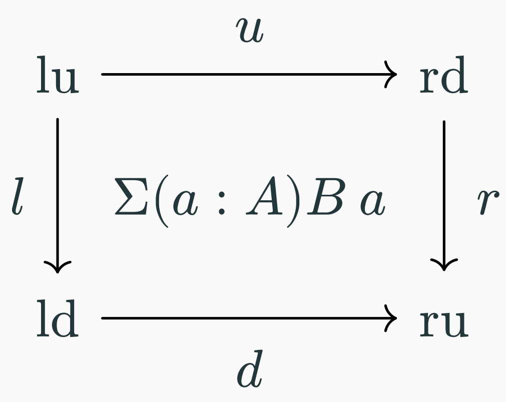

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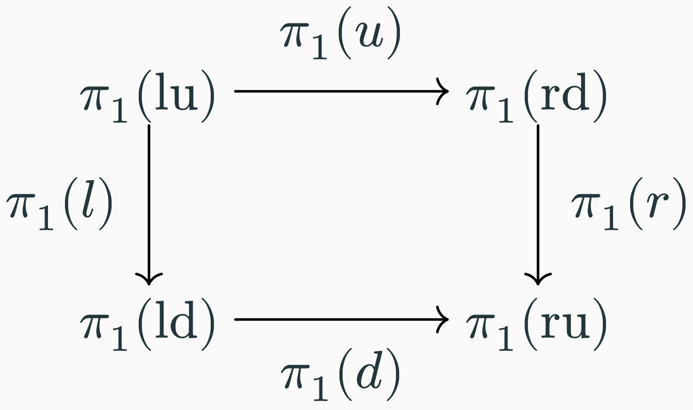

First projection in A.

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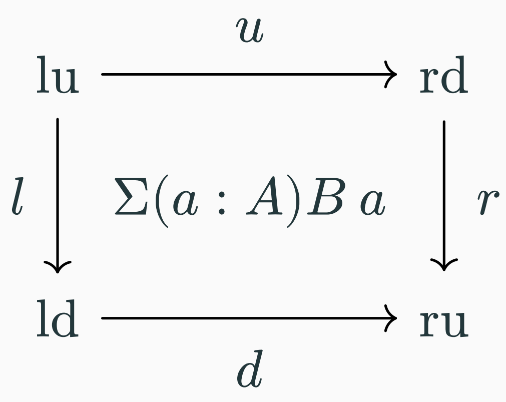

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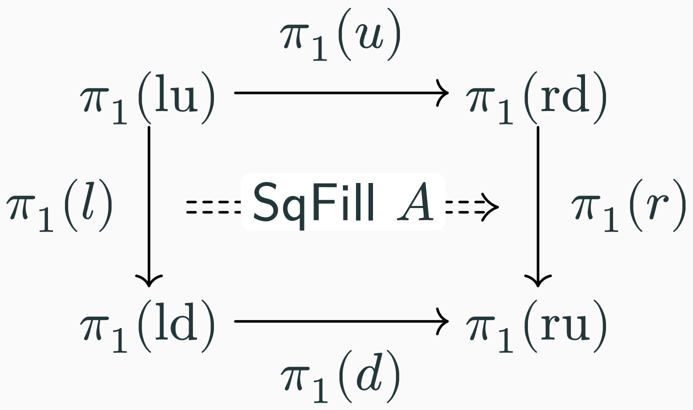

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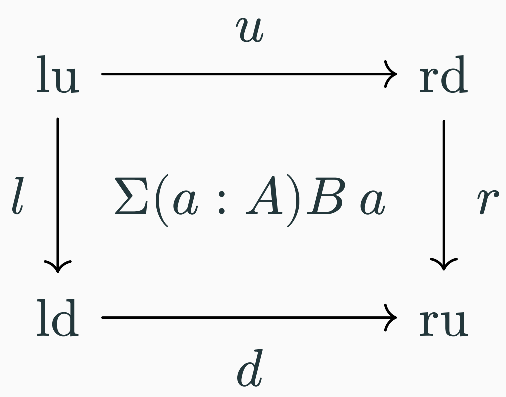
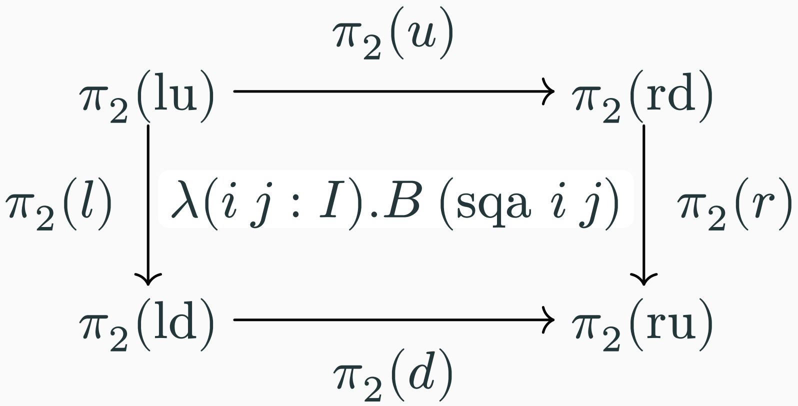


Hollow square in $\Sigma(a : A)B\ a$.



Second projection

- Theorem (SqFill-Pi): If $A : \mathrm{Type}$ and $B : A \to \mathrm{Type}$ has the SqFill property, then so does $\Pi(a : A).B\ a$. **Trivial!**
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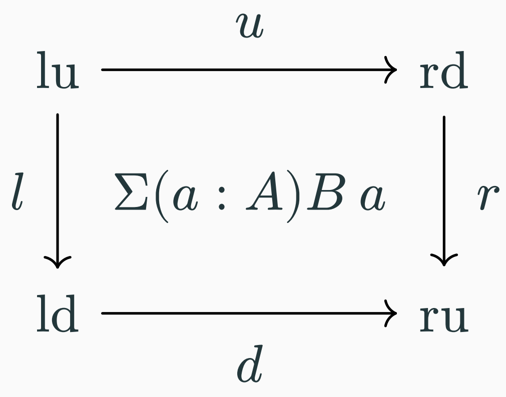

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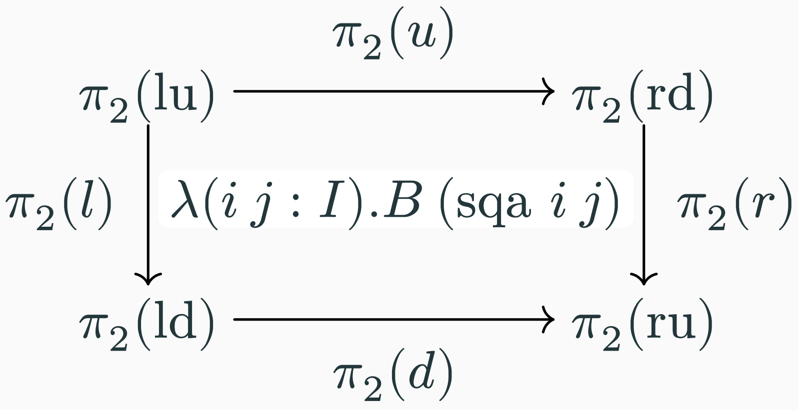

Second projection is **heterogeneous**!

How did the SqFill proofs go?

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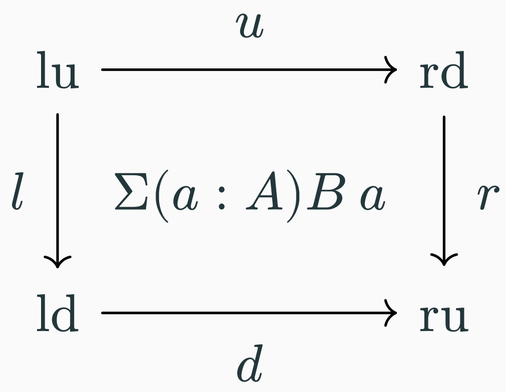

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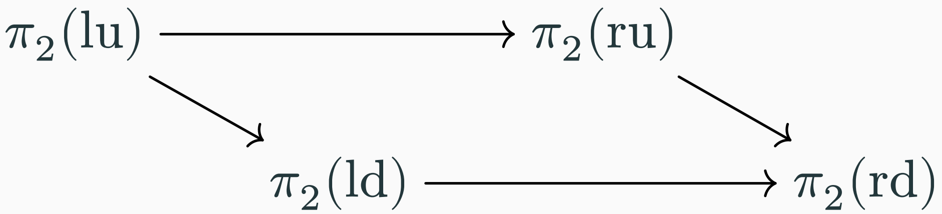

Transporting the second projection down to a fixed $B\ (\mathrm{sqa}\ i\ j)$.

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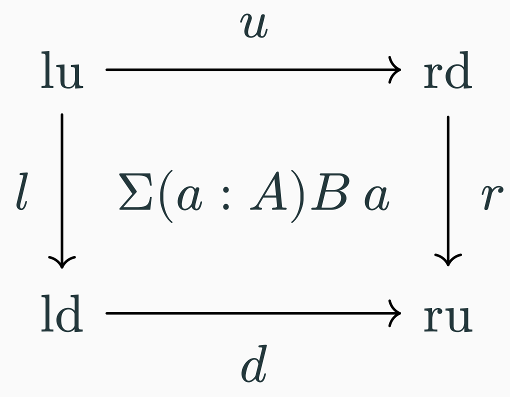

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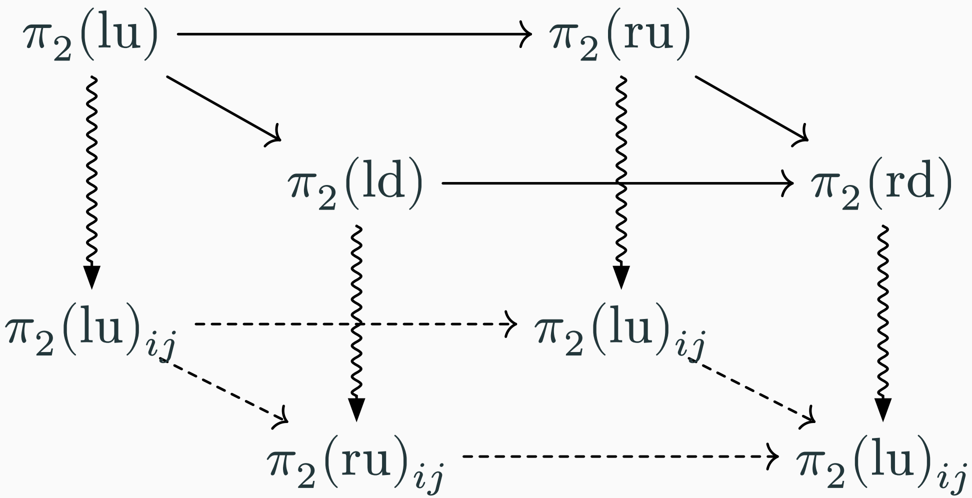

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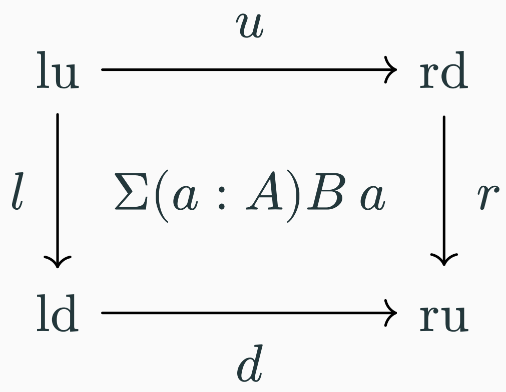

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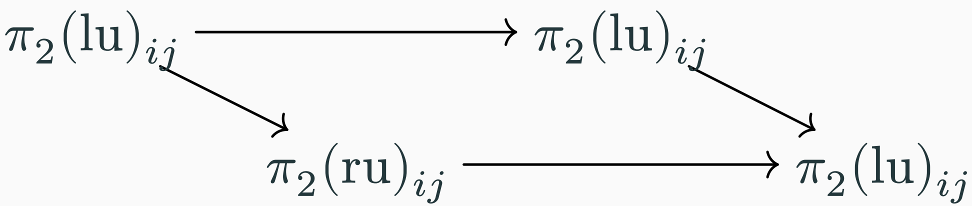

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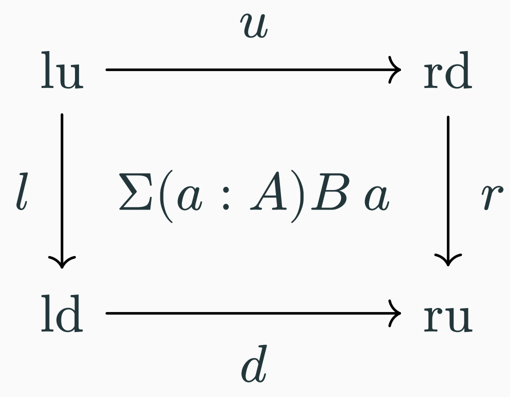

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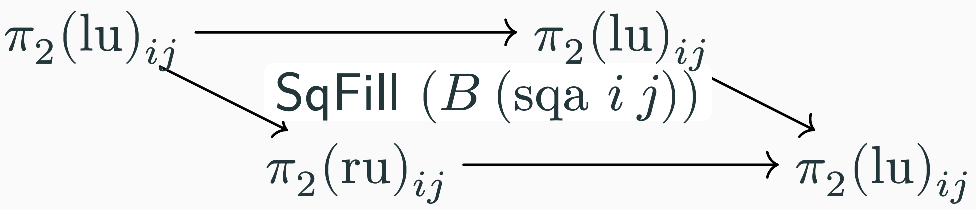

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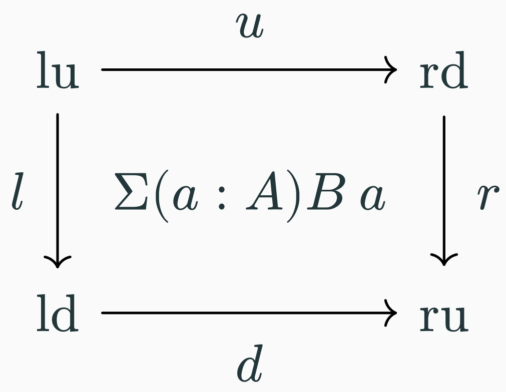

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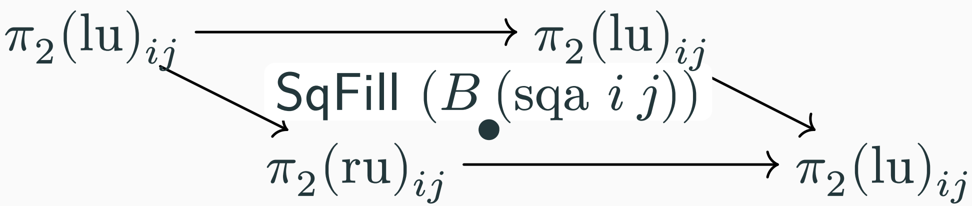

Take the (i, j)-th coordinate from the fill.

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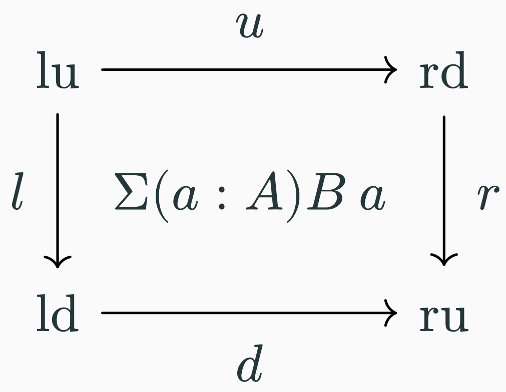

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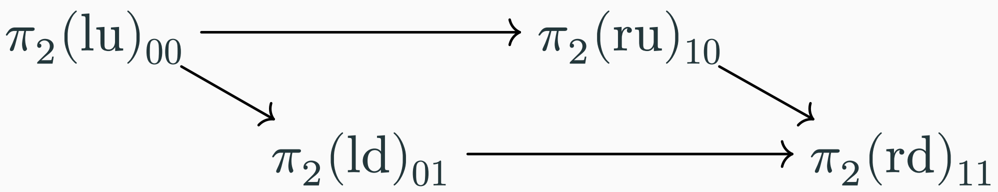

Collect all the (i, j)-th coordinates to form a **heterogeneous** square.

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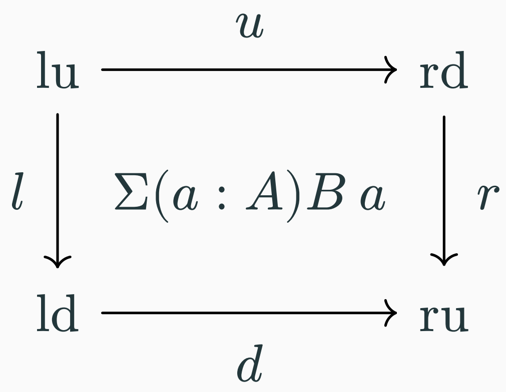

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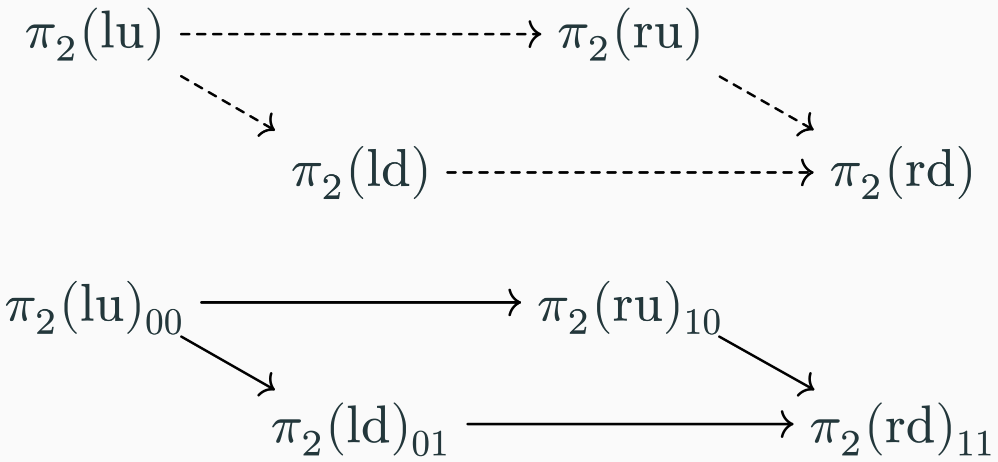

Aligning the sides of the heterogeneous square in $\Pi(i'\ j' : I).B\ (\mathrm{sqa}\ i' j')$ by comp.

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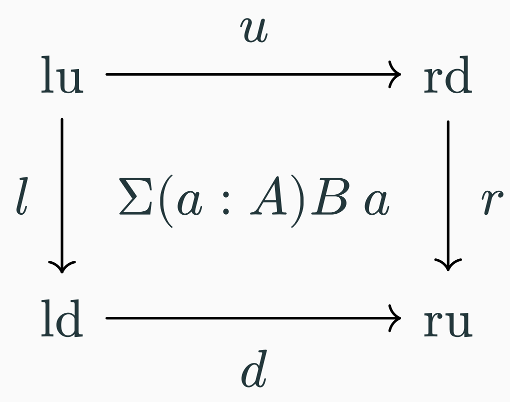

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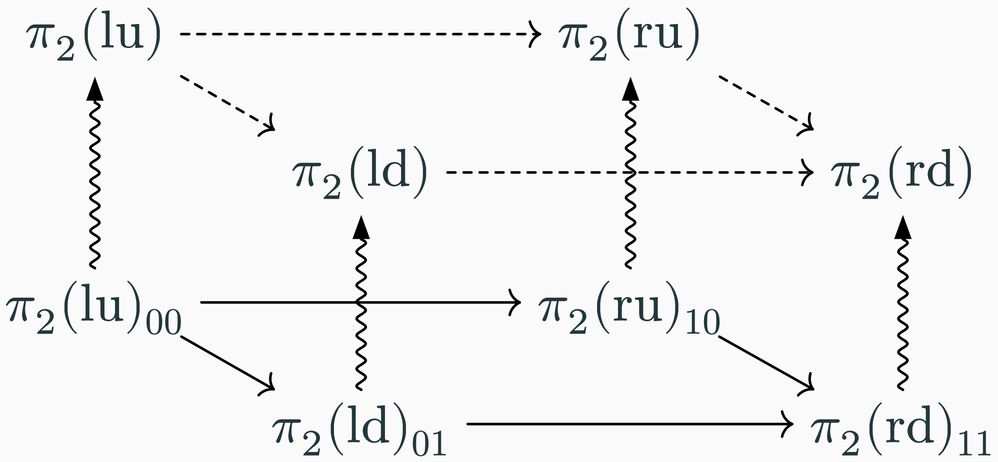

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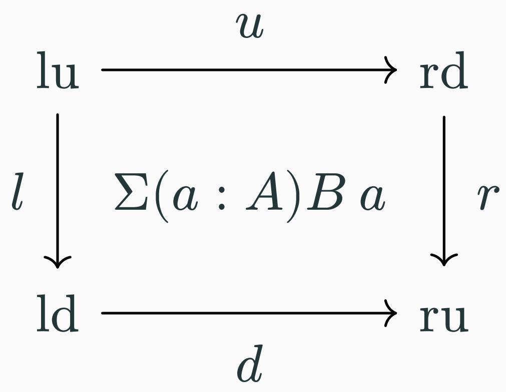

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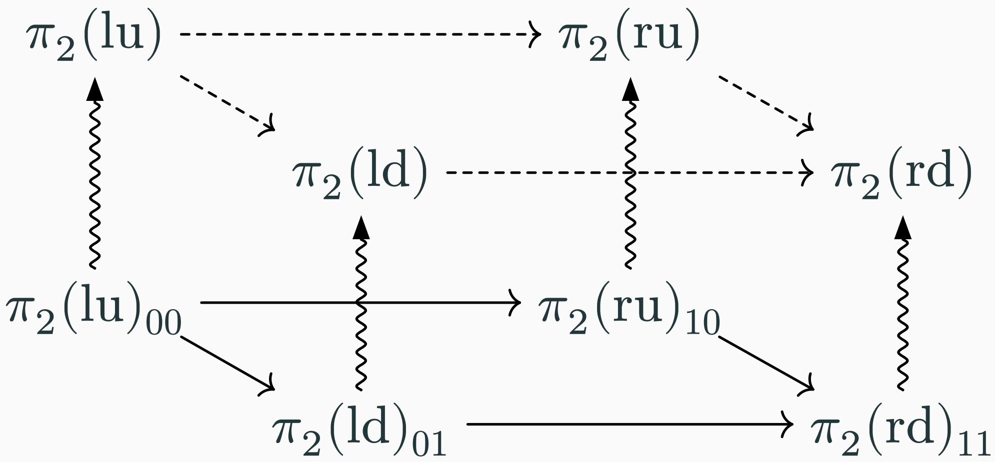

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```
if_then_else_end : I → I → I → I
if i then j else k end = (k ∧ (~ i ∨ j)) ∨ ((i ∨ k) ∧ j)

{-# INLINE if_then_else_end #-}

SqFillSigmaAB : SqFill (Σ[ a ∈ A ] B a)
SqFillSigmaAB l r u d i j .fst = SqFillA (cong fst l) (cong fst r) (cong fst u) (cong fst d) i j
SqFillSigmaAB {lu} {ld} l {ru} {rd} r u d i j .snd = outS (sqb i j)
  where
    sqa : Square (cong fst l) (cong fst r) (cong fst u) (cong fst d)
    sqa = SqFillA (cong fst l) (cong fst r) (cong fst u) (cong fst d)

    spread : (i j i' j' : I) → sqa i j ≡ sqa i' j'
    spread i j i' j' k = sqa (if k then i' else i end) (if k then j' else j end)

    lub : B (sqa i0 i0)
    lub = snd lu
    lub' : B (sqa i j)
    lub' = transport (λ k → B (spread i0 i0 i j k)) lub
    LemmaLU : PathP (λ k → B (spread i0 i0 i j k)) lub lub'
    LemmaLU k = transp (λ l → B (spread i0 i0 i j (k ∧ l))) (~ k) lub

    ldb : B (fst ld)
    ldb = snd ld
    ldb' : B (sqa i j)
    ldb' = transport (λ k → B (spread i0 i1 i j k)) ldb
    LemmaLD : PathP (λ k → B (spread i0 i1 i j k)) ldb ldb'
    LemmaLD k = transp (λ l → B (spread i0 i1 i j (k ∧ l))) (~ k) ldb

    lb : PathP (λ k → B (spread i0 i0 i0 i1 k)) lub ldb
    lb = cong snd l
    lb' : PathP (λ k → B (spread i j i j k)) lub' ldb'
    lb' j' = comp (λ k → B (spread (k ∧ i) (k ∧ j) (k ∧ i) (~ k ∨ j) j'))
                  (λ where
                    k (j' = i0) → LemmaLU k
                    k (j' = i1) → LemmaLD k) (lb j')
    LemmaL : PathP (λ k' → PathP (λ k → B (spread (k' ∧ i) (k' ∧ j) (k' ∧ i) (~ k' ∨ j) k)) (LemmaLU k')
(LemmaLD k')) lb lb'
    LemmaL k' = transport-filler (λ k' → PathP (λ k → B (spread (k' ∧ i) (k' ∧ j) (k' ∧ i) (~ k' ∨ j) k))
(LemmaLU k') (LemmaLD k')) lb k'

    rub : B (fst ru)
    rub = snd ru
    rub' : B (sqa i j)
    rub' = transport (λ k → B (spread i1 i0 i j k)) rub
    LemmaRU : PathP (λ k → B (spread i1 i0 i j k)) rub rub'
    LemmaRU k = transp (λ l → B (spread i1 i0 i j (k ∧ l))) (~ k) rub

    rdb : B (fst rd)
    rdb = snd rd
    rdb' : B (sqa i j)
    rdb' = transport (λ k → B (spread i1 i1 i j k)) rdb
    LemmaRD : PathP (λ k → B (spread i1 i1 i j k)) rdb rdb'
    LemmaRD k = transp (λ l → B (spread i1 i1 i j (k ∧ l))) (~ k) rdb

    rb : PathP (λ j → B (sqa i1 j)) rub rdb
    rb = cong snd r
    rb' : rub' ≡ rdb'
    rb' j' = comp (λ k → B (spread (~ k ∨ i) (k ∧ j) (~ k ∨ i) (~ k ∨ j) j'))
                  (λ where
                    k (j' = i0) → LemmaRU k
                    k (j' = i1) → LemmaRD k) (rb j')
    LemmaR : PathP (λ k' → PathP (λ k → B (spread (~ k' ∨ i) (k' ∧ j) (~ k' ∨ i) (~ k' ∨ j) k)) (LemmaRU k')
(LemmaRD k')) rb rb'
    LemmaR k' = transport-filler (λ k' → PathP (λ k → B (spread (~ k' ∨ i) (k' ∧ j) (~ k' ∨ i) (~ k' ∨ j) k))
(LemmaRU k') (LemmaRD k')) rb k'

    ub : PathP (λ i → B (sqa i i0)) lub rub
    ub = cong snd u
    ub' : lub' ≡ rub'
    ub' i' = comp (λ k → B (spread (k ∧ i) (k ∧ j) (~ k ∨ i) (k ∧ j) i'))
                  (λ where
                    k (i' = i0) → LemmaLU k
                    k (i' = i1) → LemmaRU k) (ub i')
    LemmaU : PathP (λ k' → PathP (λ k → B (spread (k' ∧ i) (k' ∧ j) (~ k' ∨ i) (k' ∧ j) k)) (LemmaLU k')
(LemmaRU k')) ub ub'
    LemmaU k' = transport-filler (λ k' → PathP (λ k → B (spread (k' ∧ i) (k' ∧ j) (~ k' ∨ i) (k' ∧ j) k))
(LemmaLU k') (LemmaRU k')) ub k'

    db : PathP (λ i → B (sqa i i1)) ldb rdb
    db = cong snd d
    db' : ldb' ≡ rdb'
    db' i' = comp (λ k → B (spread (k ∧ i) (~ k ∨ j) (~ k ∨ i) (~ k ∨ j) i'))
                  (λ where
                    k (i' = i0) → LemmaLD k
                    k (i' = i1) → LemmaRD k) (db i')
    LemmaD : PathP (λ k' → PathP (λ k → B (spread (k' ∧ i) (~ k' ∨ j) (~ k' ∨ i) (~ k' ∨ j) k)) (LemmaLD k')
(LemmaRD k')) db db'
    LemmaD k' = transport-filler (λ k' → PathP (λ k → B (spread (k' ∧ i) (~ k' ∨ j) (~ k' ∨ i) (~ k' ∨ j) k))
(LemmaLD k') (LemmaRD k')) db k'

    sqb-hollow : (i' j' : I) → Partial (i' ∨ j' ∨ ~ i' ∨ ~ j') (B (sqa i' j'))
    sqb-hollow i' j' (i' = i0) = l j' .snd
    sqb-hollow i' j' (i' = i1) = r j' .snd
    sqb-hollow i' j' (j' = i0) = u i' .snd
    sqb-hollow i' j' (j' = i1) = d i' .snd

    sqb'-hollow : (i' j' : I) → Partial (i' ∨ j' ∨ ~ i' ∨ ~ j') (B (sqa i j))
    sqb'-hollow i' j' (i' = i0) = lb' j'
    sqb'-hollow i' j' (i' = i1) = rb' j'
    sqb'-hollow i' j' (j' = i0) = ub' i'
    sqb'-hollow i' j' (j' = i1) = db' i'

    sqb' : (i' j' : I) → (B (sqa i j)) [ (i' ∨ j' ∨ ~ i' ∨ ~ j') ↦ sqb'-hollow i' j' ]
    sqb' i' j' = inS (SqFillB (sqa i j) lb' rb' ub' db' i' j')

    sqb : (i' j' : I) → (B (sqa i' j')) [ ( i' ∨ ~ i' ∨ j' ∨ ~ j' ) ↦ sqb-hollow i' j' ]
    sqb i' j' = inS (comp (λ k → B (spread i j i' j' k)) (
                   λ where
                     k (i' = i0) → LemmaL (~ k) j'
                     k (i' = i1) → LemmaR (~ k) j'
                     k (j' = i0) → LemmaU (~ k) i'
                     k (j' = i1) → LemmaD (~ k) i') (outS (sqb' i' j')))
```

How did the SqFill proofs go?

- Theorem (SqFill-Pi): If $A : \mathrm{Type}$ and $B : A \to \mathrm{Type}$ has the SqFill property, then so does $\Pi(a : A).B\ a$. **Trivial!**
- Theorem (SqFill-Sigma): If $A : \mathrm{Type}$ and $B : A \to \mathrm{Type}$ has the SqFill property, then so does $\Sigma(a : A).B\ a$. **Very complicated**.

```
if_then_else_end : I → I → I → I
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{-# INLINE if_then_else_end #-}

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SqFillSigmaAB {lu} {ld} l {ru} {rd} r u d i j .snd = outS (sqb i j)
  where
    sqa : Square (cong fst l) (cong fst r) (cong fst u) (cong fst d)
    sqa = SqFillA (cong fst l) (cong fst r) (cong fst u) (cong fst d)

    spread : (i j i' j' : I) → sqa i j ≡ sqa i' j'
    spread i j i' j' k = sqa (if k then i' else i end) (if k then j' else j end)

    lub : B (sqa i0 i0)
    lub = snd lu
    lub' : B (sqa i j)
    lub' = transport (λ k → B (spread i0 i0 i j k)) lub
    LemmaLU : PathP (λ k → B (spread i0 i0 i j k)) lub lub'
    LemmaLU k = transp (λ l → B (spread i0 i0 i j (k ∧ l))) (~ k) lub

    ldb : B (fst ld)
    ldb = snd ld
    ldb' : B (sqa i j)
    ldb' = transport (λ k → B (spread i0 i1 i j k)) ldb
    LemmaLD : PathP (λ k → B (spread i0 i1 i j k)) ldb ldb'
    LemmaLD k = transp (λ l → B (spread i0 i1 i j (k ∧ l))) (~ k) ldb

    lb : PathP (λ k → B (spread i0 i0 i0 i1 k)) lub ldb
    lb = cong snd l
    lb' : PathP (λ k → B (spread i j i j k)) lub' ldb'
    lb' j' = comp (λ k → B (spread (k ∧ i) (k ∧ j) (k ∧ i) (~ k ∨ j) j'))
                  (λ where
                    k (j' = i0) → LemmaLU k
                    k (j' = i1) → LemmaLD k) (lb j')
    LemmaL : PathP (λ k' → PathP (λ k → B (spread (k' ∧ i) (k' ∧ j) (k' ∧ i) (~ k' ∨ j) k)) (LemmaLU k')
(LemmaLD k')) lb lb'
    LemmaL k' = transport-filler (λ k' → PathP (λ k → B (spread (k' ∧ i) (k' ∧ j) (k' ∧ i) (~ k' ∨ j) k))
(LemmaLU k') (LemmaLD k')) lb k'

    rub : B (fst ru)
    rub = snd ru
    rub' : B (sqa i j)
    rub' = transport (λ k → B (spread i1 i0 i j k)) rub
    LemmaRU : PathP (λ k → B (spread i1 i0 i j k)) rub rub'
    LemmaRU k = transp (λ l → B (spread i1 i0 i j (k ∧ l))) (~ k) rub

    rdb : B (fst rd)
    rdb = snd rd
    rdb' : B (sqa i j)
    rdb' = transport (λ k → B (spread i1 i1 i j k)) rdb
    LemmaRD : PathP (λ k → B (spread i1 i1 i j k)) rdb rdb'
    LemmaRD k = transp (λ l → B (spread i1 i1 i j (k ∧ l))) (~ k) rdb

    rb : PathP (λ j → B (sqa i1 j)) rub rdb
    rb = cong snd r
    rb' : rub' ≡ rdb'
    rb' j' = comp (λ k → B (spread (~ k ∨ i) (k ∧ j) (~ k ∨ i) (~ k ∨ j) j'))
                  (λ where
                    k (j' = i0) → LemmaRU k
                    k (j' = i1) → LemmaRD k) (rb j')
    LemmaR : PathP (λ k' → PathP (λ k → B (spread (~ k' ∨ i) (k' ∧ j) (~ k' ∨ i) (~ k' ∨ j) k)) (LemmaRU k')
(LemmaRD k')) rb rb'
    LemmaR k' = transport-filler (λ k' → PathP (λ k → B (spread (~ k' ∨ i) (k' ∧ j) (~ k' ∨ i) (~ k' ∨ j) k))
(LemmaRU k') (LemmaRD k')) rb k'

    ub : PathP (λ i → B (sqa i i0)) lub rub
    ub = cong snd u
    ub' : lub' ≡ rub'
    ub' i' = comp (λ k → B (spread (k ∧ i) (k ∧ j) (~ k ∨ i) (k ∧ j) i'))
                  (λ where
                    k (i' = i0) → LemmaLU k
                    k (i' = i1) → LemmaRU k) (ub i')
    LemmaU : PathP (λ k' → PathP (λ k → B (spread (k' ∧ i) (k' ∧ j) (~ k' ∨ i) (k' ∧ j) k)) (LemmaLU k')
(LemmaRU k')) ub ub'
    LemmaU k' = transport-filler (λ k' → PathP (λ k → B (spread (k' ∧ i) (k' ∧ j) (~ k' ∨ i) (k' ∧ j) k))
(LemmaLU k') (LemmaRU k')) ub k'

    db : PathP (λ i → B (sqa i i1)) ldb rdb
    db = cong snd d
    db' : ldb' ≡ rdb'
    db' i' = comp (λ k → B (spread (k ∧ i) (~ k ∨ j) (~ k ∨ i) (~ k ∨ j) i'))
                  (λ where
                    k (i' = i0) → LemmaLD k
                    k (i' = i1) → LemmaRD k) (db i')
    LemmaD : PathP (λ k' → PathP (λ k → B (spread (k' ∧ i) (~ k' ∨ j) (~ k' ∨ i) (~ k' ∨ j) k)) (LemmaLD k')
(LemmaRD k')) db db'
    LemmaD k' = transport-filler (λ k' → PathP (λ k → B (spread (k' ∧ i) (~ k' ∨ j) (~ k' ∨ i) (~ k' ∨ j) k))
(LemmaLD k') (LemmaRD k')) db k'

    sqb-hollow : (i' j' : I) → Partial (i' ∨ j' ∨ ~ i' ∨ ~ j') (B (sqa i' j'))
    sqb-hollow i' j' (i' = i0) = l j' .snd
    sqb-hollow i' j' (i' = i1) = r j' .snd
    sqb-hollow i' j' (j' = i0) = u i' .snd
    sqb-hollow i' j' (j' = i1) = d i' .snd

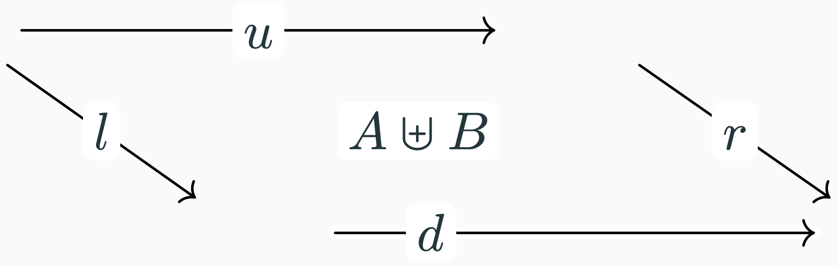
    sqb'-hollow : (i' j' : I) → Partial (i' ∨ j' ∨ ~ i' ∨ ~ j') (B (sqa i j))
    sqb'-hollow i' j' (i' = i0) = lb' j'
    sqb'-hollow i' j' (i' = i1) = rb' j'
    sqb'-hollow i' j' (j' = i0) = ub' i'
    sqb'-hollow i' j' (j' = i1) = db' i'

    sqb' : (i' j' : I) → (B (sqa i j)) [ (i' ∨ j' ∨ ~ i' ∨ ~ j') ↦ sqb'-hollow i' j' ]
    sqb' i' j' = inS (SqFillB (sqa i j) lb' rb' ub' db' i' j')

    sqb : (i' j' : I) → (B (sqa i' j')) [ ( i' ∨ ~ i' ∨ j' ∨ ~ j' ) ↦ sqb-hollow i' j' ]
    sqb i' j' = inS (comp (λ k → B (spread i j i' j' k)) (
                     λ where
                       k (i' = i0) → LemmaL (~ k) j'
                       k (i' = i1) → LemmaR (~ k) j'
                       k (j' = i0) → LemmaU (~ k) i'
                       k (j' = i1) → LemmaD (~ k) i') (outS (sqb' i' j')))
```

How did the SqFill proofs go?

- Theorem (SqFill-Coproduct): If $A\ B : \mathrm{Type}$ have the SqFill property, then so does $A \uplus B$.



Encoding an inl hollow square from $A \uplus B$ to A.

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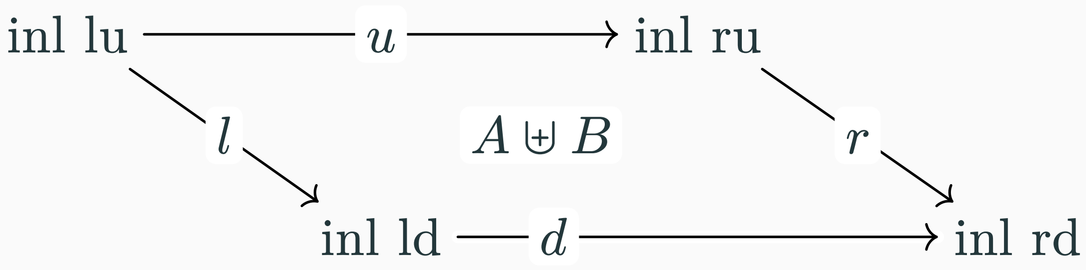

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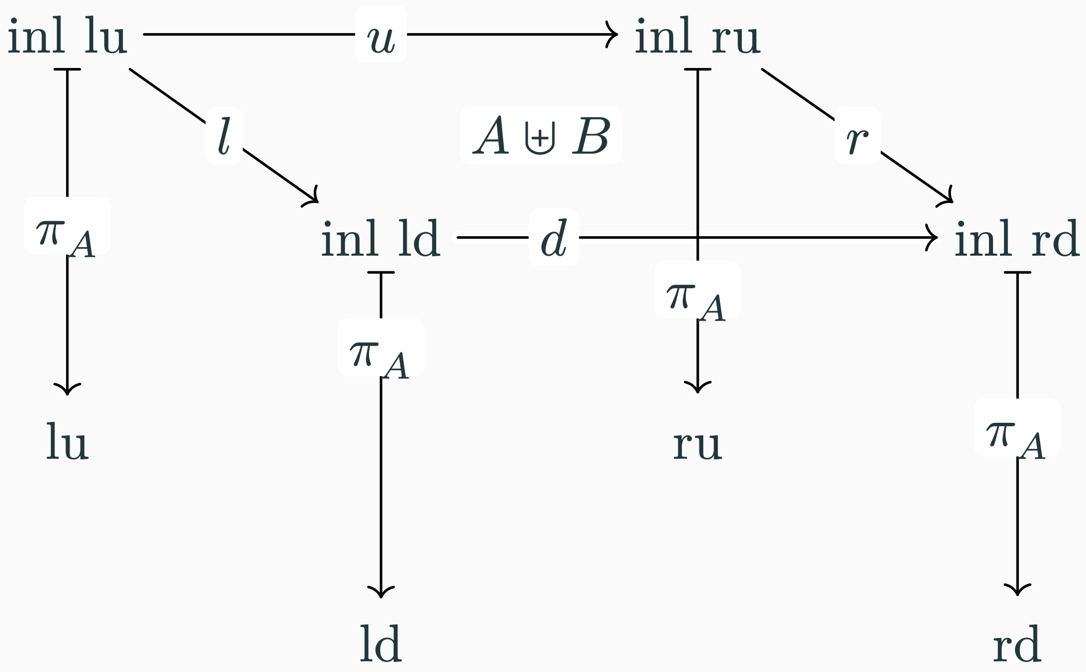

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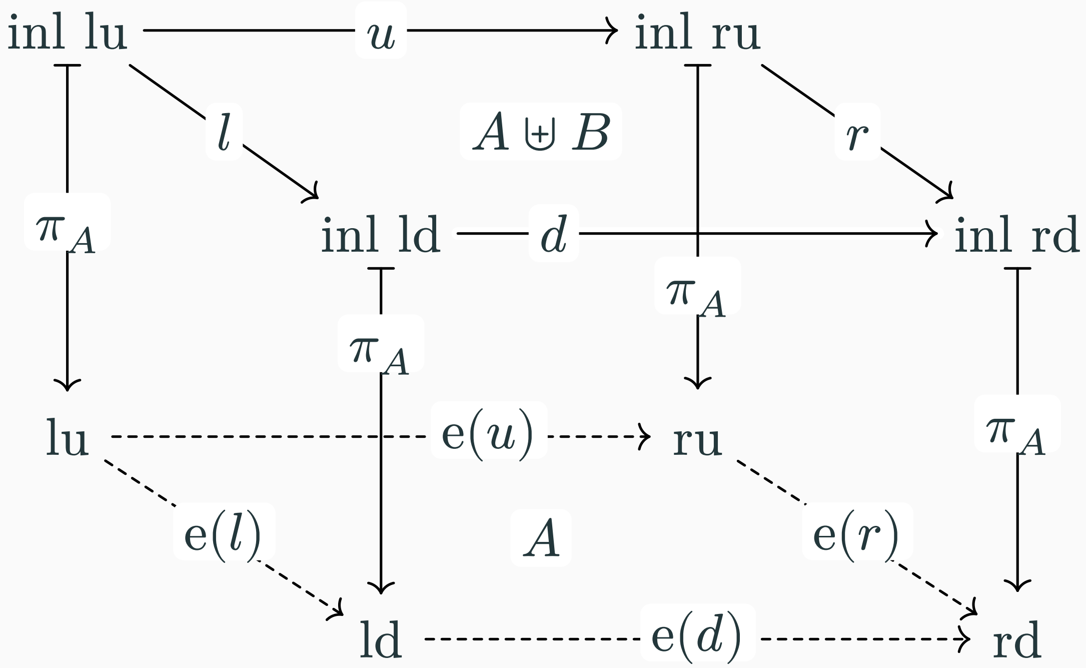

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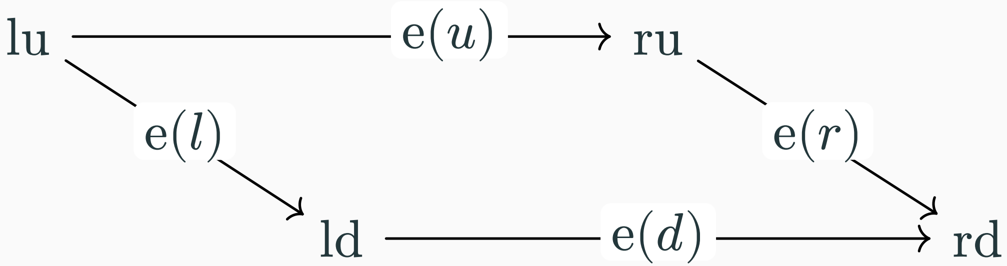

Decoding (applying inl to) the filled square in A back to $A \uplus B$.

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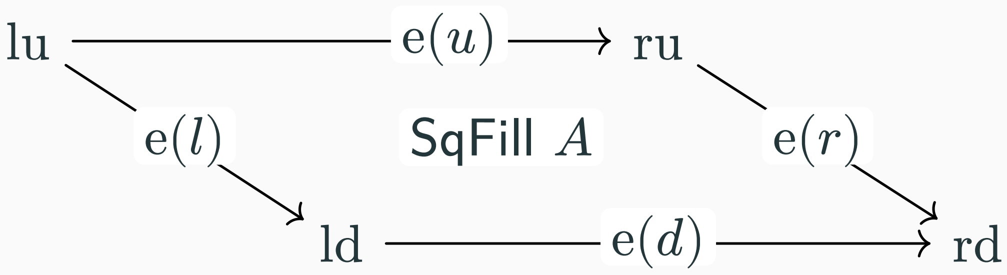

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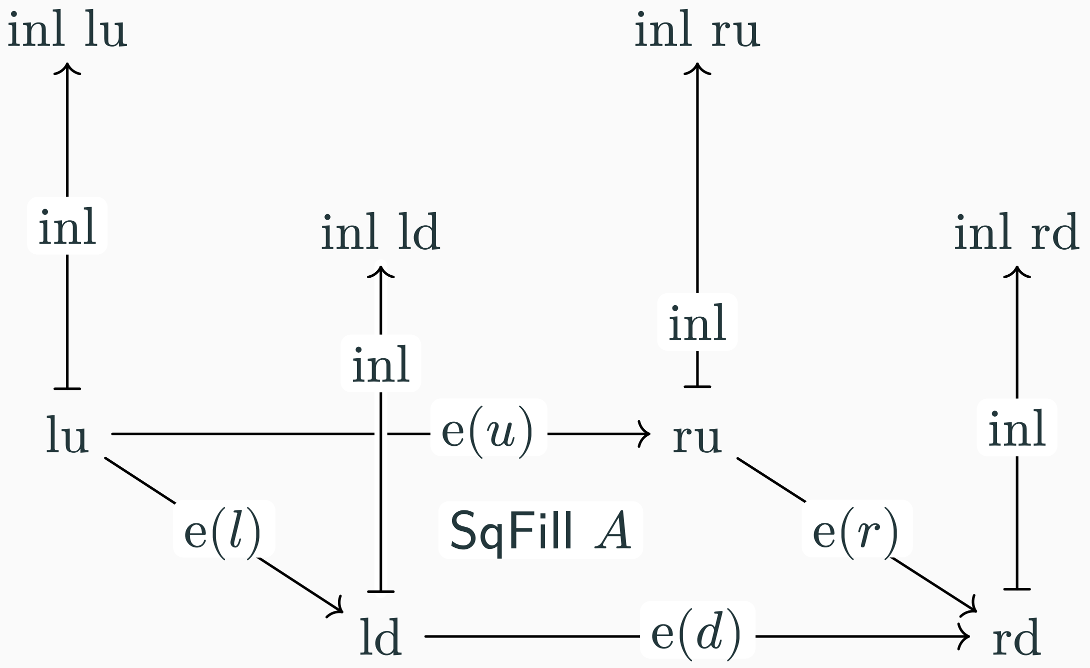

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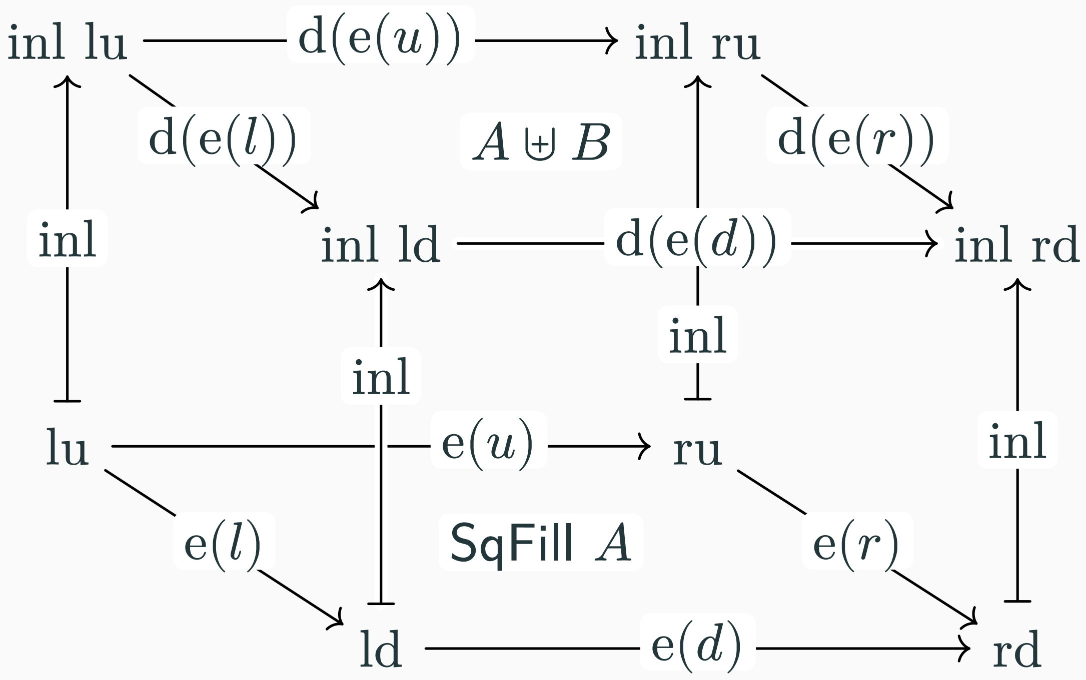

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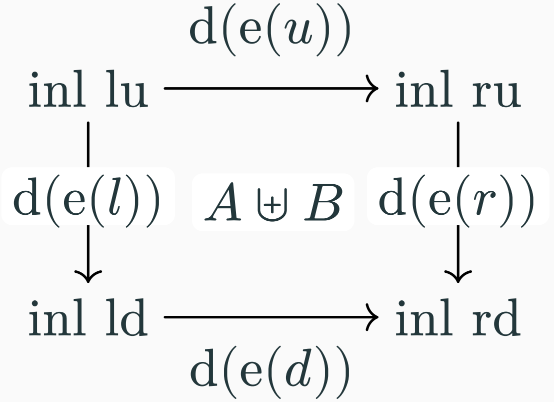

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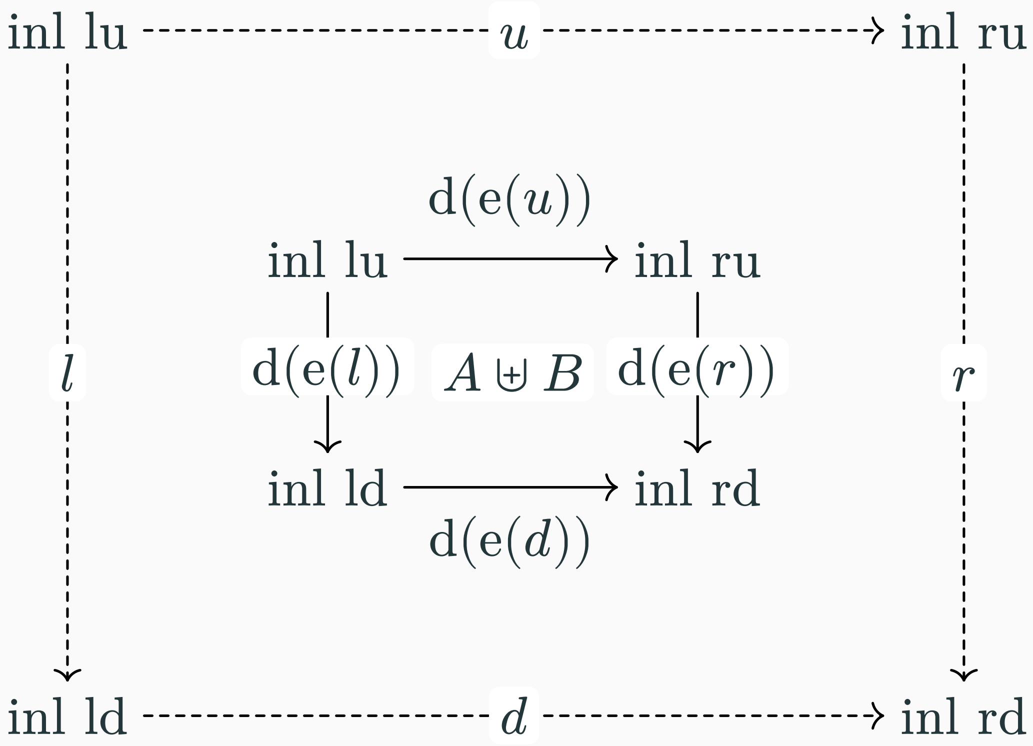

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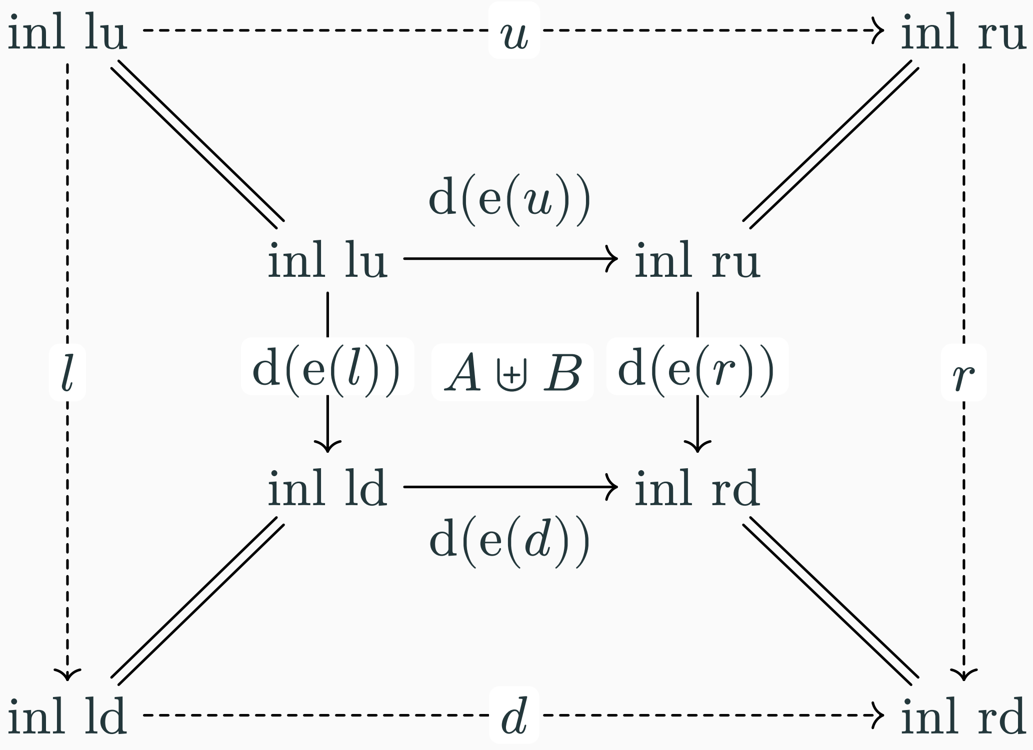

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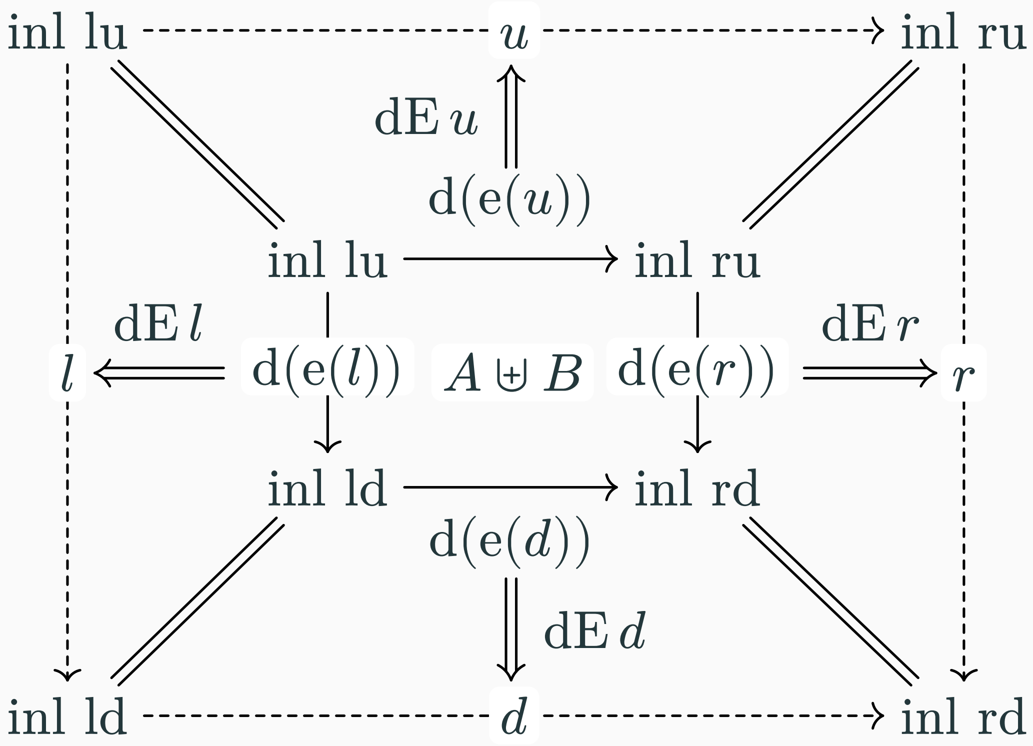

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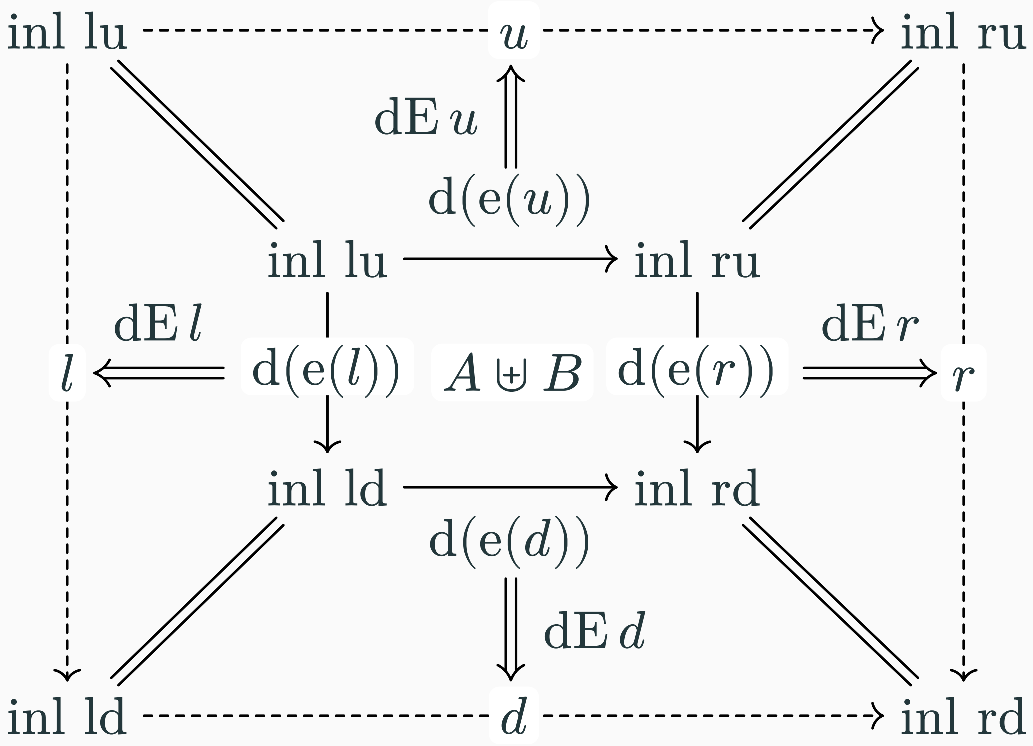

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```
Cover : {A B : Type} (c c' : A + B) → Type
Cover (inl x) (inl y) = x ≡ y
Cover (inr x) (inr y) = x ≡ y
Cover _ _ = ⊥

reflCode : {A B : Type} (c : A + B) → Cover c c
reflCode (inl x) = refl
reflCode (inr x) = refl

encode : {A B : Type} {c c' : A + B} → c ≡ c' → Cover c c'
encode {c = c} p = transport (λ i → Cover c (p i)) (reflCode c)

decode : {A B : Type} {c c' : A + B} → Cover c c' → c ≡ c'
decode {c = inl x} {c' = inl y} = cong inl
decode {c = inr x} {c' = inr y} = cong inr

decodeEncode : {A B : Type} {c c' : A + B} (p : c ≡ c') → decode (encode p) ≡ p
decodeEncode {c = inl x} = J (λ c' p → decode (encode p) ≡ p) (cong (cong inl) (transportRefl refl))
decodeEncode {c = inr x} = J (λ c' p → decode (encode p) ≡ p) (cong (cong inr) (transportRefl refl))
```

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```
SqFillCoproduct : SqFill (A + A')
SqFillCoproduct {inl lu} {inl ld} l {inl ru} {inl rd} r u d i j =
  (hcomp (λ where
      k (i = i0) → decodeEncode l k j
      k (i = i1) → decodeEncode r k j
      k (j = i0) → decodeEncode u k i
      k (j = i1) → decodeEncode d k i)
    (inl {A} {A'} (SqFillA (encode l) (encode r) (encode u) (encode d) i j)))
SqFillCoproduct {inr lu} {inr ld} l {inr ru} {inr rd} r u d i j =
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    (inr {A} {A'} (SqFillA' (encode l) (encode r) (encode u) (encode d) i j)))
      SqFillCoproduct {inl x} {inr y} l _ _ _ = ⊥-elim (inl≠inr x y l)
SqFillCoproduct {inr x} {inl y} l _ _ _ = ⊥-elim (inl≠inr y x (sym l))
SqFillCoproduct {inl x} {_} _ {inr y} _ u _ = ⊥-elim (inl≠inr x y u)
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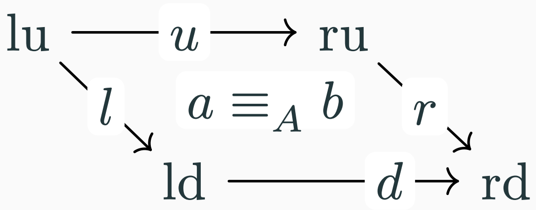
- Theorem (SqFill-Coproduct): If $A\ B : \mathrm{Type}$ have the SqFill property, then so does $A \uplus B$. Classic **encode-decode** proof [MGM04, Uni13].
- Theorem (SqFill-Path): If $A : \mathrm{Type}$ has the SqFill property, then for any $a\ b : A$, the path type $a \equiv_A b$ also has the SqFill property.



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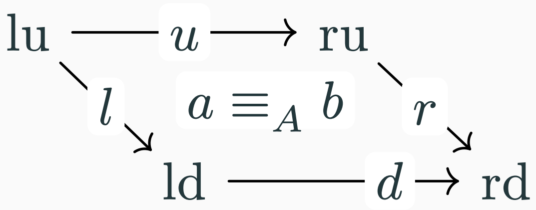

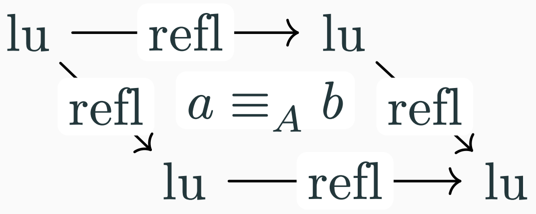

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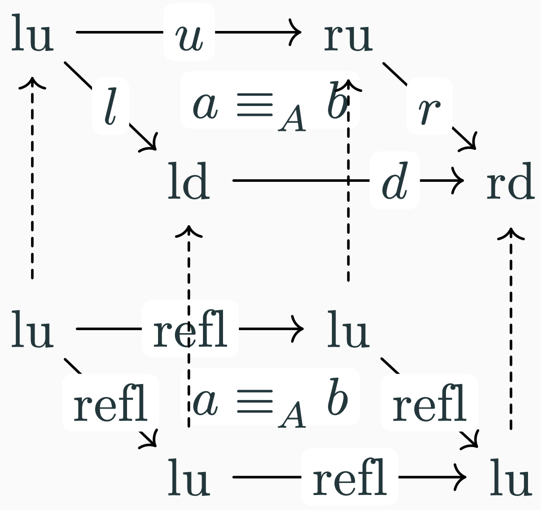

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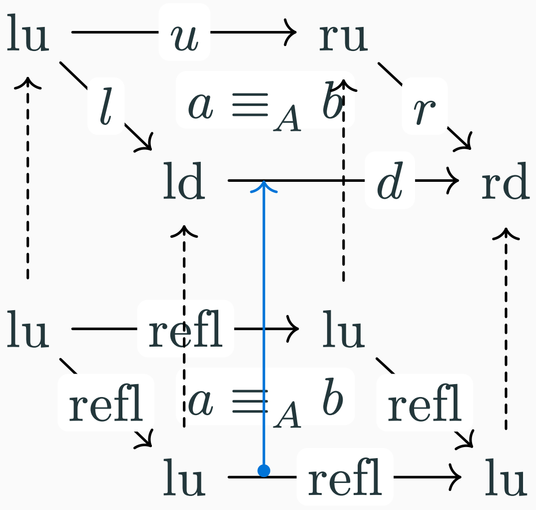

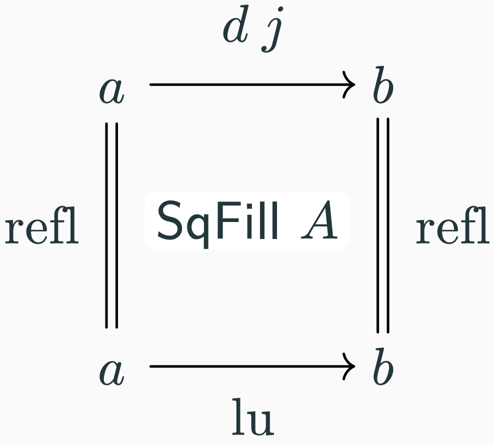

isProp: Every line on the sides is a square in A, which follows from the SqFill A assumption.

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- Theorem (SqFill-Path): If $A : \mathrm{Type}$ has the SqFill property, then for any $a\ b : A$, the path type $a \equiv_A b$ also has the SqFill property. **Simple**.

```
SqFillPath : {a b : A} → SqFill (a ≡ b)
SqFillPath {_} {_} {lu} l r u d i j =
  hcomp (λ k → λ {(i = i0) → isPropa≡b lu (l j) k
                ; (i = i1) → isPropa≡b lu (r j) k
                ; (j = i0) → isPropa≡b lu (u i) k
                ; (j = i1) → isPropa≡b lu (d i) k}) lu
  where
  isPropa≡b : {a b : A} (p q : a ≡ b) → p ≡ q
  isPropa≡b p q = SqFillA p q refl refl
```

SqFill: Summary

	SqFill (*homogeneous* Square-Filling)
Pi	Trivial (no Kan operations)
Sigma	**Complicated**: transport-fill-align
Coproducts	Standard encode-decode proof (J, hcomp)
Path Types	Simple (a single hcomp)

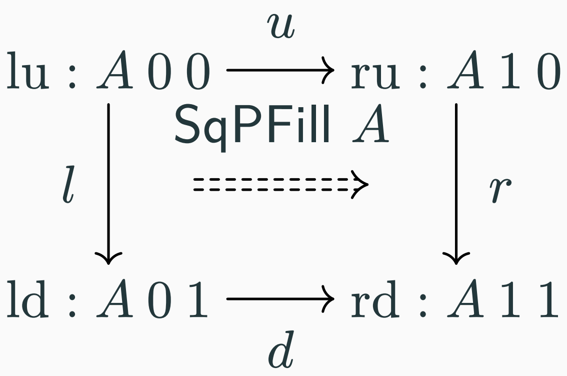
SqFill-Sigma was complicated because the second projection (dependent) was **heterogeneous**!

Generalising SqFill: SqPFill

A square of types $A : I \to I \to \mathrm{Type}$ has the **heterogeneous square-filling property** SqPFill A if the following holds:

For any hollow square in $A : I \to I \to \mathrm{Type}$, that is

- four corners $\mathrm{lu} : A\,0\,0, \mathrm{ru} : A\,1\,0, \mathrm{ld} : A\,0\,1, \mathrm{rd} : A\,0\,1$, and
- four sides connecting the four corners, namely
  1. $l : \mathrm{PathP}\ (\lambda j \to A\,0\,j)\ \mathrm{lu}\,\mathrm{ld}$
  2. $r : \mathrm{PathP}\ (\lambda j \to A\,1\,j)\ \mathrm{ru}\,\mathrm{rd}$
  3. $u : \mathrm{PathP}\ (\lambda i \to A\,i\,0)\ \mathrm{lu}\,\mathrm{ru}$
  4. $d : \mathrm{PathP}\ (\lambda i \to A\,i\,1)\ \mathrm{ld}\,\mathrm{rd}$



then the square has a filling: $\mathrm{PathP}\ (\lambda(i : I) \to \mathrm{PathP}\ (\lambda(j : I).A\,i\,j)\ (u\,i)\ (d\,i))\ l\,r$.

How did the SqPFill proofs go?

- Bad news: SqPFill-Pi is now exceedingly hard (was trivial).
  - The inverse problem of SqFill-Sigma previously: we have a *homogeneous* square to fill from a **heterogeneous** SqPFill assumption...

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- Good news: SqPFill-Sigma is **trivial**. (Great!)
- Okay news: SqPFill-Coproduct follows exactly the same way (encode-decode).
- Bad news: SqPFill-Path is now very complicated (was just one hcomp)... or a more unconventional induction hypothesis (similar to course-of-values induction)

Full proofs: clickable HTML version of Agda proofs + diagrams in report.

Summary and Observations

	SqFill (*homogeneous* Square-Filling)	SqPFill (*heterogeneous* Square-Filling)
Pi	Trivial (no Kan operations)	**Complicated**: transport-fill-align*†
Sigma	**Complicated**: transport-fill-align*	Trivial (no Kan operations)
Coproducts	Standard encode-decode proof (J, irregularity‡, hcomp)	Standard encode-decode proof (J, irregularity‡, comp)
Path Types	Simple (a single hcomp)	**Complicated**: transport-fill-align*^

The proofs can be simplified if...

* : the equality function was definable in a de Morgan algebra (specifically I) (?)

† : the $\mathrm{coe}_k(i_0, i_1)$ function needs to have eta: $\mathrm{coe}_k(i, i) = i$ at all $k : I$.

‡ : irregularity [Swa18] in CubTT (very slightly) complicates the encode-decode proof.

^ : trivial if using a stronger (course-of-values style) induction hypothesis.

Conclusion and Future Work

Conclusion

Contributions

- Implementation of a `--cubical=no-glue` variant in Agda.
- Computational UIP by "pushing through" type formers.
- SqFill and SqPFill as "generalisations" of UIP.
- Preservation proofs for Pi, Sigma, Coproduct, and Path types in `--cubical=no-glue`.

Future Work

- Preservation by inductive types (W-Types) and heterogeneous path types PathP.
  - PathP just slightly more heterogeneous than path types.
- Show nice properties of the resulting theory: Canonicity, Normalisation…
- Implement `--cubical=uip` (WIP on https://github.com/SwampertX/agda/tree/cubical-uip)
- Question: pattern matching with K on Identity types in `--cubical={no-glue/uip}`?
- Question: what constitutes a good computational rule?

Thank you!

XTT [SAG22]

- Cubical Type Theory (without Glue Types) with definitional UIP: two paths are *definitionally* equal if they have the same endpoints
- Requires a non-standard universe where type constructors are injective up to *paths*
- also possible to show the consistency of our theory by a translation into XTT.

Setoid Type Theory [Alt99, Alt+19, Hof95]

- add functional extensionality, propositional extensionality, and quotient types to intensional type theory
- "More observational" than CubTT + UIP: equality between pairs is *definitionally equal* to the pointwise equalities of the first and second components, but only an isomorphism in Cubical Type Theories.

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